UNIT 9 : DIFFERENTIABLE FUNCTIONS

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9.1 LEARNING OBJECTIVES

At the end of this unit, we will be able to :

- learn about derivative of a function and the algebra of derivatives
- know about Rolle's theorem
- know about Lagrange's mean value theorem
- know about Cauchy's mean value theorem
- learn about the Taylor's theorem with Lagrange's form of remainder
- learn about the Taylor's theorem with Cauchy's form of remainder
- learn about Taylor's series and Maclaurin's series.

9.2 INTRODUCTION

The geometrical and physical interpretation of the derivative of functions is familiar to us due to its vast applications in physics, economics, etc. In this unit, we discuss about the theory of differentiation by concentrating solely on the mathematical aspects of the derivative. In this unit, we define

the derivative of a function and discuss about the properties of derivatives. In the succeeding section, two important theorems are discussed viz. Lagrange's Mean value theorem and Cauchy's Mean value theorem. Its applications are also discussed in the section. In the last section, Taylor's theorem is discussed.

9.3 DEFINITION OF DERIVATIVES

Definition 1: Let $I \subseteq$ R be an interval, and $f : I \to$ R be a function with domain *I* and co-domain R. For any $c \in I$, a real number *L* is said to be the derivative of *f* at *c*, if, given any number $\in > 0,$ there exist a number $\delta(\in) > 0$ such that for any $x \in I$,

$$\left|\frac{f(x)-f(c)}{x-c} - L\right| < \in \text{ whenever } 0 < |x-c| < \delta(\in)$$

In this case, *f* is said to be differentiable at *c* and we denote $f'(c)$ for *L*.

In other words, we define derivative of *f* at *c* as follows :

Definition 2 : Let $I \subseteq$ R be an interval, then the function $f : I \to$ R is said to be derivable at *c* if

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

exists.

In this case, we write $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

Note: $\lim_{x \to c-} \frac{f(x)-f(c)}{x-c}$ is called the left-hand derivative of *f* at *c* and $\lim_{x \to c+} \frac{f(x)-f(c)}{x-c}$ is called the right-hand derivative of *f* at *c*. Thus the derivative $f'(c)$ exists when left-hand derivative and right-hand derivative of *f* at *c* exists and are both equal.

Example : If $f(x) := x^2$ for $x \in$ R, then for any $c \in$ R, find $f'(c)$.

We have for any $c \in$ R

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x-c} = \lim_{x \to c} (x + c) = 2c$$

Thus f' is defined for all points in R and $f'(c) = 2c$ for $c \in$ R.

The following theorem establishes that the continuity of *f* at a point *c* is a necessary condition (but not sufficient condition) for the existence of

the derivative at *c*.

Theorem : If $f : I \to \mathrm{R}$ has a derivative at $c \in I,$ then *f* is continuous at *c*

Proof : Let *f* has a derivative at *c*.

So $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Now, we can write

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c), (x \neq c)$$

$$\Rightarrow \lim_{x \to c} \{f(x) - f(c)\} = \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c}(x - c) \right\}$$

$$= \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c} \right\} \lim_{x \to c} (x - c)$$

$$= f'(c).0 = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

Thus the function *f* is continuous at *x* = *c*.

Note: The continuity of *f* at a point does not assure the existence of the derivative at that point.

Example: The function $f(x) = |x| \forall x \in \mathrm{R}$ is continuous at *x* = 0 but not differentiable at *x* = 0.

Since $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 0 = f(0)$

so *f*(*x*) is continuous at *x* = 0.
But we have,

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x|}{x}$$

$$= \lim_{x \to 0-} \frac{-x}{x} = -1$$

and

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|}{x}$$

$$= \lim_{x \to 0+} \frac{x}{x} = 1$$

Thus $\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} \neq \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$

Hence the function is not derivable at *x* = 0.



CHECK YOUR PROGRESS 1

Q 1: Show that the function *f* on R defined by

$$f(x)=\begin{cases} x \text{ if } 0\le x<1 \\ 1 \text{ if } x\ge 1 \end{cases}$$

Q 2: Show that the function $f(x)=|x-a|$ for all $x\in$ R is continuous but not derivable at $x = a$

Q 3: Show that the function *f* defined as $f(x)=\begin{cases} x^2\sin(1/x), \text{ if } x\neq 0 \\ 0 \text{ if } x=0 \end{cases}$

is derivable at $x = 0$. Also prove that $\lim_{x\to 0} f'(x)\neq f'(0)$

9.4 ALGEBRA OF DERIVATIVES

In this section, we look at the basic algebra of derivatives which are very useful in the calculation of derivatives of various combinations of functions.

Theorem : Let $f: I\to$ R and $g: I\to$ R be functions that are differentiable at *c*, where *I* : = [*a, b*] and $c\in I$. Then,

(a) For any $\alpha\in$ R, the function αf is differentiable at *c* and $(\alpha f)'(c)=\alpha f'(c)$

(b) The function $f\pm g$ is differentiable at *c* and $(f\pm g)'(c)=f'(c)\pm g'(c)$

(c) The function *fg* is differentiable at *c* and $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$

(d) For $g(c)\neq 0,$ the function *f/g* is differentiable at *c* and

$$\left(\frac{f}{g}\right)'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}$$

Proof : (a) We have $\lim_{x\to c}\dfrac{\alpha f(x)-\alpha f(c)}{x-c}=\lim_{x\to c}\dfrac{\alpha\{f(x)-f(c)\}}{x-c}$

$$=\alpha\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$=\alpha f'(c)$$

Hence for any $\alpha\in$ R, the function αf is differentiable at *c* and $(\alpha f)'(c)=\alpha f'(c)$

(b) We have

$$\lim_{x\to c}\frac{(f\pm g)(x)-(f\pm g)(c)}{x-c}=\lim_{x\to c}\frac{f(x)\pm g(x)-f(c)\mp g(c)}{x-c}$$

$$=\lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}\pm\frac{g(x)-g(c)}{x-c}\right\}$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\pm\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$$=f'(c)\pm g(c)$$

Hence function $f\pm g$ is differentiable at *c* and $(f\pm g)'(c)=f'(c)\pm g'(c)$

(c) We have

$$\lim_{x\to c}\frac{fg(x)-fg(c)}{x-c}=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}g(x)+f(c)\frac{g(x)-g(c)}{x-c}\right\}$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}g(x)+\lim_{x\to c}f(c)\frac{g(x)-g(c)}{x-c}$$

$$=f'(c)g(c)+f(c)g'(c)$$

Hence function *fg* is differentiable at *c* and $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$

(d) Let $p:=f/g$ As *g* is differentiable at *c*, so it is continuous at *c* i.e. $\lim_{x\to c}g(x)=g(c).$ Since $g(c)\neq 0$ (given) implies $\lim_{x\to c}g(x)\neq 0$ i.e. either $\lim_{x\to c}g(x)>0$ or $\lim_{x\to c}g(x)<0.$ Thus if $\lim_{x\to c}g(x)>0,$ then there exists an interval $J_1\subseteq I$ with $c\in J_1$ such that $g(x)=0.$ Similarly, if $\lim_{x\to c}g(x)<0,$ then there exists an interval $J_2\subseteq I$ with $c\in J_2$ such that $g(x)<0$. In either case we get $g(x)\neq 0$.

So we have,

$$\lim_{x\to c}\frac{(f/g)(x)-(f/g)(c)}{x-c}=\lim_{x\to c}\frac{(f(x)/g(x))-(f(c)/g(c))}{x-c}$$

$$=\lim_{x\to c}\frac{(f(x)g(c)-(f(c)g(x)}{g(x)g(c)(x-c)}$$

$$=\lim_{x\to c}\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}$$

$$=\lim_{x\to c}\frac{1}{g(x)g(c)}\left\{\frac{f(x)-f(c)}{x-c}g(c)-f(c)\frac{g(x)-g(c)}{x-c}\right\}$$

$$= \frac{1}{g(c)\lim_{x\to c} g(x)} \lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}g(c) - f(c)\frac{g(x)-g(c)}{x-c}\right\}$$

$$= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Hence for $g(c) \neq 0,$ the function *f/g* is differentiable at *c* and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Corollary : If $f_1, f_2, ..., f_n$ are functions on an interval *I* to R that are differentiable at $c \in I,$ then

1. the function $f_1 \pm f_2 \pm ... \pm f_n$ is differentiable at *c* and
$$(f_1 \pm f_2 \pm ... \pm f_n)'(c) = f_1'(c) \pm f_2'(c) \pm ... \pm f_n'(c)$$
2. the function $f_1 f_2 ... f_n$ is differentiable at *c* and

$$(f_1 f_2 ... f_n)'(c) = f_1'(c) f_2(c) ... f_n(c) + f_1(c) f_2'(c) ... f_n(c)$$
$$+ ... + f_1(c) f_2(c) ... f_n'(c)$$

3. for $f_1 = f_2 = ... = f_n = f, (f^n)'(c) = n(f(c))^{n-1} f'(c)$

Theorem : Caratheodory's Theorem: Let $f : I \to$ R such that $I := [a, b].$ Then *f* is differentiable at $c \in I$ if and only if there exists a function ϕ on *I* that is continuous at *c* and satisfies $f(x) - f(c) = \phi(x)(x - c)$ for $x \in I.$ In this case, $\phi(c) = f'(c).$

Proof : Let us suppose *f* is differentiable at *c* i.e. $f'(c)$ exists.

We defined $$\phi(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c}, & \text{for } x \neq c \\ f'(c), & \text{for } x = c \end{cases}$$

Clearly ϕ is continuous function and $\lim_{x\to c} \phi(x) = f'(c)$

If *x* = *c*, then the result follows.

If $x \neq c,$ we multiply $\phi(x)$ by $x - c$ to get *f*(*x*) – *f*(*c*) = $\phi(x)(x - c)$

Conversely, suppose there exist ϕ which is continuous at *c* and $f(x) - f(c) = \phi(x)(x - c)$

Now for $x \neq c$, we have

$$\phi(x) = \frac{f(x)-f(c)}{(x-c)}$$

$$\Rightarrow \lim_{x\to c}\phi(x) = \lim_{x\to c}\frac{f(x)-f(c)}{(x-c)}$$

But ϕ is continuous, so we have

$$\lim_{x\to c}\phi(x) = \phi(c) = \lim_{x\to c}\frac{f(x)-f(c)}{(x-c)}$$

i.e. *f* is differentiable at *c* and $f'(c) = \phi(c).$

Now we establish the theorem on differentiation of composite functions, also known as Chain rule, as follows :

Chain Rule: For the intervals *I, J* in R, let $g: I \to$ R and $f: J \to$ R be such that $f(J) \subset I.$ If *f* is differentiable at $c \in J$ and *g* is differentiable at *f*(*c*), then the composite function *gof* is differentiable at *c* and

$$(gof)'(c) = g'(f(c)).f'(c)$$

Proof: Since $f'(c)$ exists, so by the above Caratheodory's theorem, there exists function on ϕ *J* such that ϕ is continuous at *c* and $f(x)-f(c) = \phi(x)(x-c)$ for all $x \in J$ and $\phi(c) = f'(c).$

Also, since $g'(f(c))$ exists, so by the above Caratheodory's theorem, there exists function ξ on *I* such that ξ is continuous at *f*(*c*) and $g(y)-g(d) = \xi(y)(y-d)$ for all $y \in I$ and $\xi(d) = g'(d).$

Thus substituting $y = f(x)$ and $d = f(c),$ we get

$$g(f(x))-g(f(c)) = \xi(f(x))(f(x)-f(c)) = [(\xi of(x)).\phi(x)](x-c)$$

for all $x \in J$ such that $f(x) \in I.$

Again since the function $(\xi of).\phi$ is continuous at *c* and its value at *c* is $g'(f(c)).f'(c)$, so by the above Caratheodory's theorem, we get

$$(gof)'(c) = g'(f(c)).f''(c).$$

Theorem: If *f* is derivable at *c* and $f(c) \neq 0,$ then the function 1/*f* is also derivable at *c* and

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$$

Proof : We have

$$\left(\frac{1}{f}\right)'(c) = (hof)'(x), \text{ where } h(y) = 1/y \text{ for } y \neq 0$$

$$= h'(f(x))f'(x) \text{ by the Chain rule}$$

$$= \frac{f'(x)}{(f(x))^2} \text{ since } h'(y) = -1/y^2$$

9.5 DARBOUX'S THEOREM

In the succeeding sections, we establish some important theorems related with the derivatives of function. We start the section by definition of increasing and decreasing functions:

Definition: A function *f* is said to be increasing function in the interval [*a, b*] if

$$f(x_2) \geq f(x_1),\ \forall x_2 \geq x_1,\ x_1,\ x_2 \in [a,b]$$

Definition: A function *f* is said to be decreasing function in the interval [*a, b*] if

$$f(x_2) \leq f(x_1),\ \forall x_2 \geq x_1,\ x_1,\ x_2 \in [a,b]$$

Remark: The function *f* is said to be strictly increasing or strictly decreasing in [*a, b*] if

$$f(x_2) > f(x_1),\ \forall x_2 \geq x_1,$$
$$f(x_2) < f(x_1),\ \forall x_2 \geq x_1,\ x_1,\ x_2 \in [a,b]$$

Definition: A function is said to be monotonic in an interval [*a, b*] if it is either increasing or decreasing in the interval.

Let us suppose the function *f* is derivable at *x* = *c* i.e.

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c),\ x \neq c$$

Thus for any $\in > 0,$ there exist $\delta > 0$ such that

$$\left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \in \text{ whenever } 0 < |x-c| < \delta(\in)$$

i.e. $f'(c) - \in < \frac{f(x)-f(c)}{x-c} < f'(c) + \in,$ whenever $x \in (c-\delta,\ c+\delta),\ x \neq c$

Now, for $\in < f'(c),$ we obtain $f'(c) - \in > 0$ and hence

$$\frac{f(x)-f(c)}{x-c} > 0, \text{ whenever } x \in (c-\delta,\ c+\delta),\ x \neq c$$

Thus we have for $0 < \in < f'(c)$

i. $f(x) > f(c)$ for $x \in (c, c+\delta)$

ii. $f(x) < f(c)$ for $x \in (c-\delta, c)$

i.e. the function is increasing at *c* if $f'(c) > 0.$ Similarly, the function is decreasing at *c* if $f'(c) < 0$

Note: At theend point *a*, there exist an interval $(a, a+\delta)$ such that

i. $f'(a) > 0 \Rightarrow f(x) > f(a)$ for $x \in (a, a+\delta)$

ii. $f'(a) < 0 \Rightarrow f(x) < f(a)$ for $x \in (a, a+\delta)$

Note: At the end point *b*, there exist an interval $(b-\delta, b)$ such that

i. $f'(b) > 0 \Rightarrow f(x) < f(b)$ for $x \in (b-\delta, b)$

ii. $f'(b) < 0 \Rightarrow f(x) > f(b)$ for $x \in (b-\delta, b)$

We now state the Darboux's theorem as follows :

Darboux's Theorem : If a function *f* is derivable on a closed interval [*a, b*] and $f'(a), f'(b)$ are of opposite signs, then there exists at least one point *c* between *a* and *b* such that $f'(c) = 0$.

Proof: Without loss of generality, let us assume $f'(a) < 0$ and $f'(b) > 0$.

Since $f'(a) < 0$, so there exists a number $\delta_1 > 0$ such that

$$f(x) < f(a) \ \forall x \in (a, a+\delta_1) \qquad (1)$$

Also since $f'(b) > 0$, so there exists $\delta_2 > 0$ such that

$$f(x) < f(b) \ \forall x \in (b-\delta_2, b) \qquad (2)$$

Also since *f* is derivable in [*a, b*], it is continuous in [*a, b*].

But we know from the previous chapter, that continuous functions in closed interval are bounded and attains its bounds. So there exists a point $c \in$ [*a, b*] such that $f(c) = m$ where *m* is the infimum of the function *f*.

It is clear from (1) and (2), that *c* cannot coincide with *a* or *b* since otherwise there would exist point where the values of the function would be less than *f*(*c*). Thus *c* is an interior point of [*a, b*].

Now we prove that $f'(c) = 0$

If $f'(c) < 0$, then there exists an interval $(c, c+\delta_3), \delta_3 > 0$ such that for all $x \in (c, c+\delta_3)$, $f(x) < f(c) = m$, which is a contradiction since *m* is the infimum of *f*.

Similarly, if $f'(c) > 0$, then there exists an interval $(c-\delta_4, c), \delta_4 > 0$ such that for all $x \in (c-\delta_4, c)$, $f(x) < f(c) = m$, which is again a contradiction.

Hence the only possibility is $f'(c) = 0$.

9.6 ROLLE'S THEOREM

In this section, we establish the Rolle's Theorem as follows :

Rolle's Theorem : If a function *f* is (i) continuous on a closed interval

[*a, b*], derivative f' exists at every point of (*a, b*) and (iii) $f(a) = f(b),$ then there exists at least one point *c* in (*a, b*) such that $f'(c) = 0.$

Proof: Since the function is continuous on a closed interval [*a, b*], it is bounded and attains its bounds i.e. for infimum *m* and supremum *M* of the function *f*, there exists points *c* and *d* in [*a, b*] such that

$$f(c) = m \text{ and } f(d) = M$$

There are two possibilities: either *m* = *M* or $m \neq M$

If *m* = *M*, then *f* is constant function and therefore its derivatives

$$f'(x) = 0 \forall x \in [a,b]$$

If $m \neq M,$ then at least one of these, say *m* is different from $f(a)(= f(b))$ i.e.

$$f(c) = m \neq f(a) \Rightarrow c \neq a$$

$$f(c) = m \neq f(b) \Rightarrow c \neq b$$

This means $c \in (a,b)$

We now show that $f'(c) = 0$

If $f'(c) < 0,$ then tere exists an interval $(c, c + \delta_1), \delta_1 > 0$ for every point *x* of which, $f(x) < f(c) = m,$ which is a contradiction since *m* is the infimum of *f*.

Similarly, if $f'(c) > 0,$ then there exists an interval $(c - \delta_2, c), \delta_2 > 0$ for every point *x* of which, $f(x) < f(c) = m,$ which is again a contradiction.

Hence the only possibility is $f'(c) = 0.$

9.7 THE MEAN VALUE THEOREM

In this section, we discuss about Mean Value theorem and some of its applications. The Mean Value theorem relates the values of a function to the values of its derivatives. At first we state the first Mean Value theorem viz. Lagrange's Mean Value Theorem and later state the second Mean Value Theorem viz. Cauchy's Mean Value theorem.

Lagrange's Mean Value Theorem : If *f* is continuous on a closed interval [*a, b*] and *f* has a derivative in (*a, b*), then there exists at least one point *c* in (*a, b*) such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof: We consider a function

$$\phi(x) = f(x) + Ax, \ x \in [a,b]$$

where *A* is a constant to be determined such that $\phi(a) = \phi(b)$

Therefore

$$\phi(a) = \phi(b)$$
$$\Rightarrow f(a) + Aa = f(b) + Ab$$
$$\Rightarrow A = -\frac{f(b) - f(a)}{b - a}$$

Also, since the function $\phi(x)$ is a sum of two continuous and derivable function, so $\phi(x)$ is :

(i) continuous on [*a, b*]

(ii) derivable on [*a, b*]

(iii) $\phi(a) = \phi(b)$

Thus, by Rolle's theorem, there exists a real number $c \in [a,b]$ such that $\phi'(c) = 0$.

But

$$\phi'(x) = f'(x) + A$$

Thus $\phi'(c) = 0 = f'(x) + A$

$$\Rightarrow f'(c) = -A$$
$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or $f(b) - f(a) = f'(c)(b - a)$

Cauchy's Mean Value Theorem : Let *f* and *g* be two functions on [*a, b*], such that, they are (i) continuous on [*a, b*], (ii) derivable on (*a, b*), and (iii) $g'(x) \neq 0, \ \forall x \in (a,b)$. Then there exists at least one point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: If $g(a) = g(b),$ then by Rolle's theorem, *g* being continuous and derivable on [*a, b*], we would get $g'(x) = 0, \ \forall x \in (a,b)$, which is a contradiction.

Thus, we note that $g(a) \neq g(b).$

Now, consider the function

$$\phi(x) = f(x) + Ag(x),\ x \in [a,b],$$

where *A* is a constant to be determined such that $\phi(a) = \phi(b)$. Thus

$$\phi(a) = \phi(b)$$

$$\Rightarrow f(a) + Ag(a) = f(b) + Ag(b)$$

$$\Rightarrow \quad A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Also, since the function $\phi(x)$ is a sum of two continuous and derivable function, so $\phi(x)$ is :

(i) continuous on [*a, b*]

(ii) derivable on [*a, b*]

(iii) $\phi(a) = \phi(b)$

Thus, by Rolle's theorem, there exists a real number $c \in [a,b]$ such that $\phi'(c) = 0.$

But

$$\phi'(x) = f'(x) + Ag'(x)$$

Thus $$\phi'(c) = 0 = f'(c) + Ag'(c)$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = -A$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Example: Prove that

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}$$

Here $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in [\alpha, \beta]$

Also, $f'(x) = \cos x$ and $g'(x) = -\sin x$

Since *f* and *g* are both continuous and differentiable, therefore by Cauchy's Mean Value Theorem on $[\alpha, \beta],$ we have

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta},\ \alpha < \theta < \beta$$

$$\Rightarrow \frac{\sin a - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta,\ \alpha < \theta < \beta$$



CHECK YOUR PROGRESS 2

Q 1: Prove that

$$\frac{\upsilon-u}{1+\upsilon^2} < \tan^{-1}\upsilon - \tan^{-1}u < \frac{\upsilon-u}{1+u^2}$$

if $0 < u < \upsilon$. Also deduce that

$$\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$$

9.8 TAYLOR'S THEOREM

In the previous sections, we have defined the derivative of the function *f* at *x* = *c* and denoted it as $f'(c)$. In a similar manner, we define derivative of the function f' at *x* = *c* (if exists) and denote it as $f''(c)$. Clearly, the existence of $f''(c)$ implies existence and continuity of f' in a neighbourhood of *c*. The higher order derivatives can be similarly defined and the derivative of f^{n-1} at *c* (if exists) is said to be the n^{th} derivative of *f* at *c* and is denoted as $f^n(c)$.

We state the Taylor's Theorem as follows :

Taylor's Theorem : If a function *f* defined on [*a, a* + *h*], is such that

(i) the $(n-1)^{th}$ derivative f^{n-1} is continuous on [*a, a* + *h*] and

(ii) the n^{th} derivative f^n exists on (*a, a* + *h*),

then there exists at least one real number $\theta(0<\theta<1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots$$

$$\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h) \qquad (3)$$

where *p* is a given positive number.

Proof: Condition (i) in the theorem implies that all derivatives $f', f'', \ldots, f^{n-1}$ exists and are continuous on [*a, a* + *h*].

We consider the function ϕ defined on [*a, a* + *h*] as

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f^n(x) + \ldots$$

$$\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$$

where *A* is a constant to be determined such that $\phi(a+h)=\phi(a)$.
Now

$$\phi(a+h)=\phi(a)$$

$$\Rightarrow f(a+h)+0=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...$$

$$...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p \qquad (4)$$

We observe that,

(i) the function $\phi(x)$ is continuous on [*a, a + h*] since $f, f', f'', ..., f^{n-1}$ are all continuous on [*a, a + h*]

(ii) the function $\phi(x)$ is derivable in (*a, a + h*) since $f, f', f'', ..., f^{n-1}$ and $(a+h-x)^r$ for all *r* are derivable in (*a, a + h*), and

(iii) $\phi(a+h)=\phi(a)$

Thus the function $\phi(x)$ satisfies all the conditions of Rolle's Theorem and hence there exists at least one number $\theta(0<\theta<1)$ such that $\theta'(a+\theta h)=0.$ But

$$\phi'(x)=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-Ap(a+h-x)^{p-1}$$

$$\Rightarrow 0=\phi'(a+\theta h)=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)-Aph^{n-1}(1-\theta)^{p-1}$$

$$\Rightarrow A=\frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h),\ h\neq 0,\ \theta\neq 1 \qquad (5)$$

Substituting the value of *A* from (5) in (4), we get the required result,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+...$$

$$...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

Remark : The term

$$R_n=\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

which occurs after *n* terms, is known as Taylor's remainder after *n* terms.

Remark : For *p* = 1, we get

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

which is known as the Cauchy's form of remainder.

Remark : For p = n, we get

$$R_n = \frac{h^n}{n!} f^n(a+\theta h)$$

which is known as the Lagrange's form of remainder.

Second form of Taylor's Theorem : Substituting $x = a + h$ or $h = x - a$ in (3), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots$$

$$\ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta(x-a)) \quad (6)$$

where $0 < \theta < 1.$

The remainder term after n terms can be written as

$$R_n = \frac{(x-a)^n (1-\theta)^{n-p}}{p(n-1)!} f''(c))$$

where c depends on x and $a < c < x$.

9.9 TAYLOR'S SERIES AND MACLAURIN'S SERIES

In the previous section, we observe from the equation (3) that the value $f(a + h)$ of the function at a point is approximated by a summation given in R.H.S of the equation. The remainder term R_n is used to estimate the error in approximating the function $f(a + h)$. However, a natural question arises as to how far the R.H.S of (3) correctly represents the L.H.S of (3). The question can be answered if the series

$$f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots$$

converges to $f(a + h)$.

Let $S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a),$ such that

$$f(a+h) = S_n + R_n$$

Thus if $R_n \to 0$ as $n \to \infty,$ we have

$$\lim_{n\to\infty} S_n = f(a+h)$$

i.e. the infinite series

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+...$$

converses to $f(a+h)$

Thus for a function *f* having derivatives of n^{th} order in [*a, a + h*], if Taylor's Remainder $R_n \to 0$ as $n \to \infty$, then $f(a+h)$ can be written as

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+...$$

Definition: The infinite series

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+... \qquad (7)$$

is said to be the Taylor's series.

Replacing $x=a+h,$ we get,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+... \qquad (8)$$

which is the expansion of *f*(*x*) in powers of (*x* – *a*).

Definition: For all $x \in [0,h],$ we get from (8), (by replacing *a* = 0),

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$$

which is known as the Maclaurin's infinite series expansion of *f*(*x*) in powers of *x*.

Example: Derive the expansion of e^x using Maclaurin's theorem.

Here $f(x)=e^x$ is continuous and derivable for all *x*.

Taking Lagrange's form of remainder, we get

$$R_n=\frac{x^n}{n!}f^n(\theta x)=\frac{x^n}{n!}e^{\theta x},\ 0<\theta<1$$

Thus

$$\lim_{n\to\infty} R_n=\lim_{n\to\infty}\frac{x^n}{n!}e^{\theta x}=e^{\theta x}\lim_{n\to\infty}\frac{x^n}{n!}=0$$

i.e. $R_n \to 0$ as $n \to \infty.$

Also *f*(0) = 1 and $f^n(0)=1$ for all integral value of *n*.

Thus substituting these values in Maclaurin's series, we get

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...\forall x \in \mathbb{R}$$



CHECK YOUR PROGRESS 3

Q 1: Find the expansion of sin *x* using Maclaurin's series.

Q 2: Find the expansion of cos *x* using Maclaurin's series.

Q 3: Find the expansion of log (*x* + 1) using Maclaurin's series.



9.10 LET US SUM UP

- Let $I \subseteq$ R be an interval, and $f : I \to$ R be a function with domain *I* and co-domain R. For any $c \in I$, a real number *L* is said to be the derivative of *f* at *c*, if, given any number $\in > 0$, there exist a number $\delta(\in) > 0$ such that for any $c \in I$,

  $$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \in \text{ whenever } 0 < |x - c| < \delta(\in)$$

  In this case, *f* is said to be differentiable at *c* and we denote $f'(c)$ for *L*.

- Let $I \subseteq$ R be an interval, then the function $f : I \to$ R is said to be derivable at *c* if

  $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

  exists.

  In this case, we write $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

- $\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$ is called the left-hand derivative of *f* at *c* and $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$ is called the right-hand derivative of *f* at *c*. Thus the derivative $f'(c)$ exists when left-hand derivative and right-hand derivative of *f* at *c* exists and are both equal.

- If $f : I \to$ R has a derivative at $c \in I$, then *f* is continuous at *c*.

- The continuity of *f* at a point does not assure the existence of the derivative at that point.

- Let $f : I \to \mathrm{R}$ and $g : I \to \mathrm{R}$ be functions that are differentiable at *c*, where $I := [a, b]$ and $c \in I$. Then,

  (a) For any $\alpha \in \mathrm{R}$, the function of αf is differentiable at *c* and $(\alpha f)'(c) = \alpha f'(c)$

  (b) The function $f \pm g$ is differentiable at *c* and $(f \pm g)'(c) = f(c) \pm g'(c)$

  (c) The function *fg* is differentiable at *c* and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

  (d) For $g(c) \neq 0$, the function *f/g* is differentiable at *c* and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

- If $f_1, f_2, ..., f_n$ are functions on an interval *I* to R that are differentiable at $c \in I$, then

  (a) the function $f_1 \pm f_2 \pm ... \pm f_n$ is differentiable at *c* and $(f_1 \pm f_2 \pm ... \pm f_n)'(c) = f_1'(c) \pm f_2'(c) \pm ... \pm f_n'(c)$

  (b) the function $f_1 f_2 ... f_n$ is differentiable at *c* and

$$(f_1 f_2 ... f_n)'(c) = f_1'(c) f_2(c) ... f_n(c) + f_1(c) f_2'(c) ... f_n(c) + ... + f_1(c) f_2(c) ... f_n'(c)$$

  (c) for $f_1 = f_2 = ... = f_n = f$, $(f^n)'(c) = n(f(c))^{n-1} f'(c)$

- Let $f : I \to \mathrm{R}$ such that $I := [a, b]$. Then *f* is differentiable at $c \in I$ if and only if there exists a function ϕ on *I* that is continuous at *c* and satisfies $f(x) - f(c) = \phi(x)(x - c)$ for $x \in I$. In this case, $\phi(c) = f'(c)$.
- For the intervals *I, J* in R, let $g : I \to \mathrm{R}$ and $f : J \to \mathrm{R}$ be such that $f(J) \subset I$. If *f* is differentiable at $c \in J$ and *g* is differentiable at *f*(*c*), then the composite function *gof* is differentiable at *c* and $(gof)'(c) = g'(f(c)).f'(c)$
- If *f* is derivable at *c* and $f(c) \neq 0$, then the function 1/*f* is also derivable at *c* and

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$$

- Darboux's Theorem: If a function *f* is derivable on a closed interval [*a, b*] and $f'(a), f'(b)$ are of opposite signs, then there exists at least one point *c* between *a* and *b* such that $f'(c) = 0$.

- Rolle's Theorem: If a function *f* is (i) continuous on a closed interval [*a, b*], derivative f' exists at every point of (*a, b*) and (iii) *f*(*a*) = *f*(*b*), then there exists at least one point *c* in (*a, b*) such that $f'(c) = 0.$
- Lagrange's Mean Value Theorem : If *f* is continuous on a closed interval [*a, b*] and *f* has a derivative in (*a, b*), then there exists at least one point *c* in (*a, b*) such that

  $$f(b) - f(a) = f'(c)(b - a)$$
- Cauchy's Mean Value Theorem: Let *f* and *g* be two functions on [*a, b*], such that, they are (i) continuous on [*a, b*), (ii) derivable on (*a, b*), and (iii) $g'(x) \neq 0, \ \forall x \in$ (*a, b*). Then there exists at least one point $c \in (a,b)$ such that

  $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$
- Taylor's Theorem: If a function *f* defined on $[a, a + h]$, is such that (i) the $(n-1)^{th}$ derivative f^{n-1} is continuous on [*a, a* + *h*] and (ii) the n^{th} derivative f^n exists on (*a, a* + *h*), then there exists at least one real number $\theta (0 < \theta < 1)$ such that

  $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + ...$$

  $$... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n (a + \theta h)$$

  where *p* is a given positive number.
- The term

  $$R_n = \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n (a + \theta h)$$

  which occurs after *n* terms, is known as Taylor's remainder after *n* terms.
- For *p* = 1, we get

  $$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n (a + \theta h)$$

  which is known as the Cauchy's form of remainder.
- Foro *p* = *n*, we get

  $$R_n = \frac{h^n}{n!} f^n (a + \theta h)$$

  which is known as the Lagrange's form of remainder.

- Substituting $x = a + h$ or $h = x - a$, we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

$$\ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta(x-a))$$

where $0 < \theta < 1$ and the remainder term after *n* terms can be written as

$$R_n = \frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f''(c))$$

where *c* depends on *x* and *a* < *c* < *x*.

- The infinite series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots$$

(9)

is said to be the Taylor's series.

- For all $x \in [0, h]$, the infinite series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

is known as the Maclaurin's infinite series expansion of *f*(*x*) in powers of *x* which is obtained by replacing *a* = 0 in the Taylor's series.

9.11 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Ans to Q No 1: At *x* = 1, we have

$$\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x-1}{x-1} = 1$$

and $$\lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{1-1}{x-1} = 0$$

Since right hand derivative $\neq$ left hand derivative, so $f'(1)$ does not exist.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Check your progress 2

Ans to Q No 1: Try yourself.

Check your progress 3

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.



9.12 FURTHER READING

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7) Bartle, R.G., Sherbert, D. R. (2011). Introduction to real analysis. Hoboken, NJ: Wiley.
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9.13 MODEL QUESTIONS

Q 1: Show that the function $f(x)=|x|+|x-1|$ for all $x \in \mathrm{R}$ is continuous but not derivable at $x = 0$ and $x = 1$.

Q 2: Let f be a twice differentiable function such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one ξ between a and b for which $f''(\xi) < 0$.

Q 3: Find the expansion of $(1+x)^m$ using Maclaurin's series, for all $x \in \mathrm{R}$, where m is any integer.

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UNIT 10 : FUNCTIONS OF SEVERAL VARIABLES

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Structure of R^n

10.4 Explicit and implicit Functions

10.5 Limit and Continuity of Function from R^n to R

10.6 Directional Derivatives

10.7 Partial Derivatives

10.8 Let Us Sum Up

10.9 Answers to Check Your Progress

10.10 Further Reading

10.11 Model Questions

10.1 LEARNING OBJECTIVES

At the end of this unit, we will be able to :

- know about the structure of R^n
- learn about the limit and continuity of functions from R^n to R
- learn about the directional derivatives and partial derivatives.

10.2 INTRODUCTION

In the previous units, we studied about functions which depend on a single variable *x*. But in real-world phenomenon, a quantity being investigated usually depends on two or more independent variables. So we need to extend the basic ideas of functions of a single variable to functions of several variables. The derivatives of functions of several variables are more varied and more interesting because of the different ways in which the variables can interact. There is a vast application of these functions of several variables in probability theory, statistics, fluid dynamics, etc.

10.3 STRUCTURE OF R^n

From the previous units, we have observed that the real number system R is more than just a collection of points. It has various algebraic, and order structures. The same is true of many other important sets of mathematical objects. Among the most important are the euclidean set R^n which we study in this chapter. The collection of elements of the set R^n is in the form of *n*-tuples of real numbers which we define as follows :

$$x = (x_1, x_2, ..., x_n)$$

Definition: The set R^n is defined as the set containing n-tuples $x = (x_1, x_2, ..., x_n)$ of real numbers. That is, R^n is the Cartesian product of R given as :

$R^n = R \times R \times ... \times R$ (*n* times)

The element $x \in R^n$ are called the points of R^n. They can also be viewed as vectors, and R as a field of scalars. We define addition of vectors in R^n as coordinate-wise addition:

If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, then

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \qquad (1)$$

For $\alpha \in R$ and $x = (x_1, x_2, ..., x_1) \in R^n$ we define

$$\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n) \qquad (2)$$

We refer to such multiplication of a vector by a scalar as scalar multiplication. With these notions of addition and scalar multiplication, R^n possesses the algebraic structure of a vector space.

Theorem 1 : If addition and scalar multiplication on R^n are defined by (1) and (2), respectively, then the following are true for all $x, y, z \in R^n$ for all $\alpha, \beta, \in R$:

(a) $x + y$ and $\alpha x \in R^n$ (closure of R^n under addition and scalar multiplication)

(b) $x + y = y + x$ (commutative law)

(c) $(x + y) + z = x + (y + z)$ (associative law)

(d) $x + 0 = x$, where 0 = (0, 0, ..., 0)

(e) $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$ (distributive law)

(f) $\alpha(\beta x) = (\alpha\beta)x$